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Reflection of sine-Gordon breathers

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The influence of a boundary on a breather traveling in a Josephson line cavity is examined by means of numerical computations. For a passive termination the breather is reflected into a breather of less energy; when the characteristic impedance of the line equals the external load resistor the breather is almost annihilated. At an active termination the breather is reflected into a breather of increased energy possibly followed by fluxons, antifluxons, or even breathers.

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I. INTRODUCTION

The dynamics of fluxons on Josephson-junction transmission lines has been studied extensively in recent years. In particular, we mention the papers by McLaughlin and Scott[1] and Kaup and Newell[2] which contain many references. In Refs. 1 and 2 a perturbation theory for fluxons propagating on Josephson lines with bias, inpurities and losses is formulated. Some of the results from this theory were compared with numerical solutions by Christainsen and Olsen.[3] Costabile *et al.*[4] have given an exact solution to the sine-Gordon equation without perturbing terms for a finite line with open-end boundary conditions. For a finite line with fixed-end boundary conditions corresponding to a short-circuited line, DeLeonardis *et al.*[5] obtained exact solutions using the same method. A numerical investigation of the reflection of a single fluxon on a semi-infinite Josephson line at a passive or an active boundary has been performed by Christiansen and Olsen.[6] They found a single reflected fluxon, an antifluxon, absorption of the incident fluxon, or fission into an arbitrary number of fluxons depending on the boundary conditions. Microwave oscillators based on the resonant propagation of fluxons were investigated by Erné and Parmentier.[7]

Recently the bion or breather solution to the sine-Gordon equation has attracted considerable interest.[8-11] The breather can be considered as a bound state of a fluxon and an antifluxon. Furthermore, fluxons, antifluxons, and breathers possess the remarkable soliton property. Unlike the fluxons and the antifluxons, the breather does not require an activation energy because its rest energy can range from 0 to $2E_0$, where E_0 is the rest energy of the fluxons or the antifluxons. Further, breathers have an oscillatory degree of freedom, which increases their physical potential. For a finite line Costabile *et al.* have also found exact breather solutions.[12]

In the present paper the effect of a boundary on a breather is examined by means of numerical computations. For a passive termination of the Josephson line we find that the breather is reflected into a breather with decreased energy. The breather can almost be annihilated at a certain choice of the boundary condition. The paper is structured as follows: In Sec. II we review the model formulation. Section III contains our numerical results. In Sec. IV we show the change of energy as function of the boundary condition and comment on the results.

II. MODEL FORMULATION

In Ref. 13 it is shown that the transverse voltage v and the surface current i on a Josephson line can be expressed in terms of the phase difference ϕ by

$$\phi_x = -(2eL/\hbar)i, \tag{2.1}$$

$$\phi_t = (2e/\hbar)v. \tag{2.2}$$

Here L is the inductance per length unit of the line, e is the electronic charge, and $\hbar$ is Planck's constant, and ϕ satisfies the sine-Gordon equation

$$\phi_{xx} - LC\phi_{tt} = (2eLJ_0/\hbar)\sin\phi, \tag{2.3}$$

where C is the capacitance per length unit and J_0 is the Josephson current per length unit for the line.

So far, only boundary conditions corresponding to an open-ended line ($i = 0 \Rightarrow \phi_x = 0$) and a short-circuited line ($v = 0 \Rightarrow \phi_t = 0$) have been considered in the literature. In fact, the exact solutions obtained by Perring and Skyrme[14] for the fluxon-antifluxon collision and the fluxon-fluxon collision may be viewed as a reflection at a boundary placed at the collision point with boundary conditions $\phi_x = 0$ and $\phi_t = 0$, respectively. Similar two-breather solutions exist because of the soliton property. Thus a breather is reflected without loss of energy when the boundary conditions are $\phi_x = 0$ or $\phi_t = 0$.

For the loaded termination of the left end of the Josephson line the boundary condition becomes

$$v(0,t) = -Ri(0,t), \tag{2.4a}$$

or

$$\phi_x(0,t) - (L/R)\phi_t(0,t) = 0. \tag{2.4b}$$

Here the termination is passive when the resistance $R > 0$. This boundary condition models the practical application of the Josephson line as a generator of electromagnetic (em) radiation.[4] In the present paper we shall also consider the case where $R < 0$. This corresponds to an active termination of the Josephson line. [For the case of a right-end termination of the Josephson line the minus sign in Eq. (2.4) is replaced by a plus sign.]

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Introduction of dimensionless variables $x(2eLJ_0/\hbar)^{\frac{1}{2}} \curvearrowright x$ and $t\,[2eJ_0/(\hbar C)]^{\frac{1}{2}} \curvearrowright t$ changes Eqs. (2.3) and (2.4b) into

$$\phi_{xx} - \phi_{tt} = \sin\phi \tag{2.5}$$

and

$$\phi_x(0,t) - \alpha\phi_t(0,t) = 0, \tag{2.6}$$

where we have introduced the parameter

$$\alpha = \left(\frac{L}{C}\right)^{1/2} \Big/ R. \tag{2.7}$$

Thus α is the ratio of the characteristic impedance of the line $(L/C)^{1/2}$ to the external load resistor R. For $\alpha = 0$ and $\alpha = \infty$, Eq. (2.6) reduced to $\phi_x = 0$ and $\phi_t = 0$, respectively.

III. NUMERICAL RESULTS

In order to study the behavior of a single breather in a Josephson-line cavity of dimensionless length l we solve the initial value problem

$$\begin{aligned} &\phi_{xx} - \phi_{tt} = \sin\phi \\ &\phi_x(0,t) - \alpha\phi_t(0,t) = 0, \\ &\phi_x(l,t) + \alpha\phi_t(l,t) = 0, \\ &\phi(x,0) = F(x,0), \\ &\phi_t(x,0) = F_t(x.0), \end{aligned} \tag{3.1}$$

where the boundary conditions at $x = 0$ and $x = l$ are left-end and right-end versions of Eqs. (2.6), and

$$F(x,t) = 4\arctan\left(\tan\theta\,\frac{\sin[\gamma(u)\cos\theta\,(t - ux - t_0)]}{\cosh[\gamma(u)\sin\theta\,(x - ut - x_0)]}\right),$$

$$\gamma(u) = (1 - u^2)^{-1/2}, \tag{3.2a}$$

and (3.2b)

$$\tan\theta = (1 - \omega^2)^{1/2}/\omega. \tag{3.2c}$$

The initial condition (3.2) correspond to a breather initially placed at $x = x_0$ traveling with the velocity u. The parameter θ determines the initial amplitude and the terms $\gamma(u)\sin\theta$ and $\gamma(u)\cos\theta$ determine the initial width and frequency of oscillation, respectively. The parameter t_0 is a phase constant.

The numerical results are obtained by means of a computer program based on the method of characteristics and displayed in terms of $\phi_x(x,t)$.

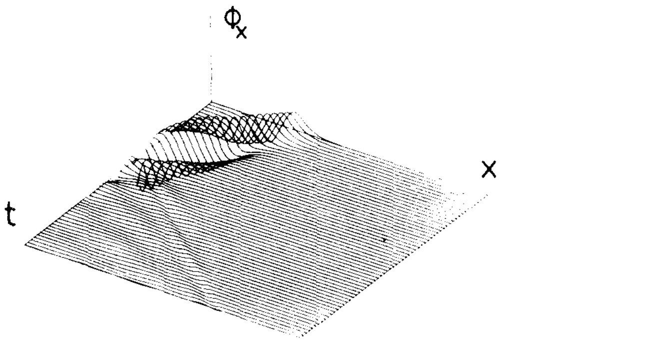


FIG. 2. The incident breather is almost annihilated at the left end of a finite line. Numerical solution of Eq. (3.1) with $l = 30$, $\alpha = 1$, $u = -0.5$, $\tan\theta = 2$, $x_0 = 7.5$, $t_0 = 0$, and $0 \leqslant t \leqslant 36$. Only a minor part is reflected into radiation.

Figure 1 shows the propagation of a breather. The parameters in Eq. (3.1) are chosen to be $l = 30$, $\alpha = 0.1$, $u = -0.5$, $\tan\theta = 2$, $x_0 = 7.5$, $t_0 = 0$, and $0 \leqslant t \leqslant 48$. At the passive termination the breather is reflected into a breather of less energy. From the numerical results we find for the reflected breather $\tan\theta_{\rm refl} \simeq 1.11$ and $u_{\rm refl} \simeq 0.22$. The reflection is seen to give rise to the creation of radiation.

In Fig. 2 we show an incident breather which is almost annihilated at the left end of a finite line. Numerical solution of Eq. (3.1) with $l = 30$, $\alpha = 1$, $u = -0.5$, $\tan\theta = 2$, $x_0 = 7.5$, $t_0 = 0$, and $0 \leqslant t \leqslant 36$. Only a minor part is reflected into radiation.

Figure 3 shows the propagation of a breather which is reflected at the left end of a finite line with a passive termination. The parameters in Eq. (3.1) are chosen to be $l = 30$, $\alpha = 20$, $u = -0.5$, $\tan\theta = 2$, $x_0 = 7.5$, $t_0 = 0$, and $0 \leqslant t \leqslant 36$. At the passive termination the breather is reflected into a breather of less energy. The numerical results show that $\tan\theta_{\rm refl} \simeq 1.28$ and $u_{\rm refl} \simeq 0.5$ for the reflected breather. In this case hardly any radiation is observed.

In order to investigate the influence of t_0 we have in Fig. 4 changed t_0 from 0 to 3.04 compared with the parameters in the former figure. Thus the parameters in Eq. (3.1) are $l = 30$, $\alpha = 20$, $u = -0.5$, $\tan\theta = 2$, $x_0 = 7.5$, $t_0 = 3.04$, and $0 \leqslant t \leqslant 36$. From the numerical results we find no depen-

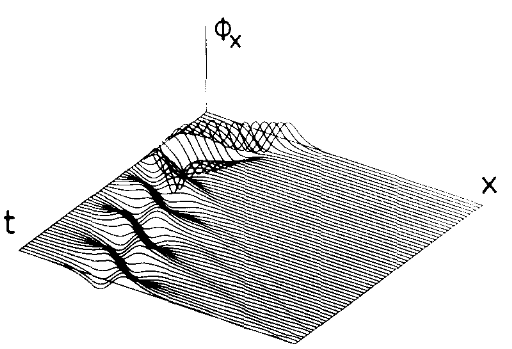


FIG. 1. Reflection of a breather at the left end of a finite line. Numerical solution of Eq. (3.1) with $l = 30$, $\alpha = 0.1$, $u = -0.5$, $\tan\theta = 2$, $x_0 = 7.5$, $t_0 = 0$, and $0 < t < 48$. At this passive termination the breather is reflected into a breather of less energy. As a result of the reflection radiation is observed.

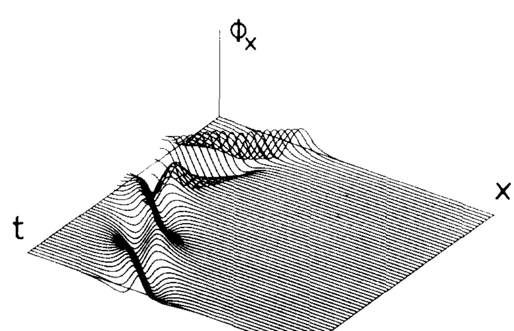


FIG. 3. Reflection of a breather at the left end of a finite line. Numerical solution of Eq. (3.1) with $l = 30$, $\alpha = 20$, $u = -0.5$, $\tan\theta = 2$, $x_0 = 7.5$, $t_0 = 0$ and $0 < t < 36$. At the passive termination the breather is reflected into a breather of less energy. Hardly any radiation is observed as a result of the reflection.

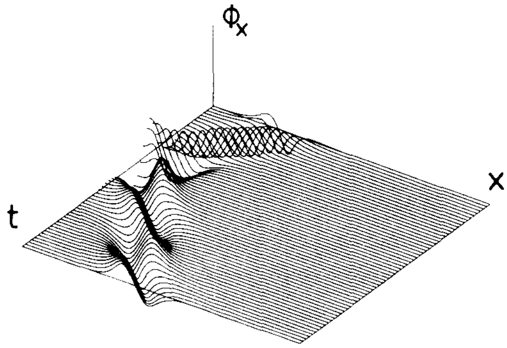


FIG. 4. Reflection of a breather at the left end of a finite line. Numerical solution of Eq. (3.1) with $l = 30$, $\alpha = 20$, $u = -0.5$, $\tan\theta = 2$, $x_0 = 7.5$, $t_0 = 3.04$, and $0 \leqslant t \leqslant 36$. The only change in parameters compared with the former figure is a change in t_0. The numerical results show no dependence of the frequency and velocity on t_0.

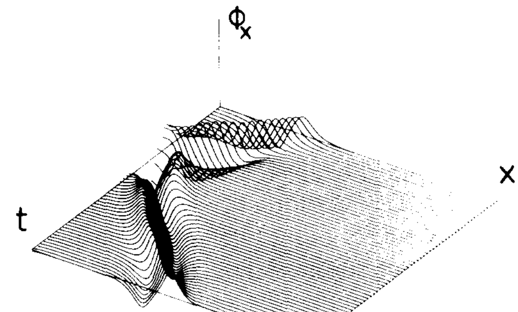


FIG. 6. Reflection of a breather at the left end of a finite line. Numerical solution of Eq. (3.1) with $l = 30$, $\alpha = -20$, $u = -0.5$, $\tan\theta = 2$, $x_0 = 7.5$, $t_0 = 0$, and $0 \leqslant t \leqslant 36$. At the active termination the breather is reflected into a breather of increased energy.

dence of the frequency and velocity on t_0.

Figure 5 shows the propagation of a breather which is reflected at the left end of a finite line with an active termination. The parameters in Eq. (3.1) are chosen to be $l = 30$, $\alpha = -0.1$, $u = -0.5$, $\tan\theta = 2$, $x_0 = 7.5$, $t_0 = 0$, and $0 \leqslant t \leqslant 36$. At the active termination the breather is reflected into a breather of increased energy. From Fig. 5 it is seen that the reflected breather is almost separated into a fluxon followed by an antifluxon.

Finally, Fig. 6 shows the reflection of a breather at an active termination. The parameters in Eq. (3.1) are $l = 30$, $\alpha = -20$, $u = -0.5$, $\tan\theta = 2$, $x_0 = 7.5$, $t_0 = 0$, and $0 \leqslant t \leqslant 36$. The breather is reflected into a breather of increased energy.

For α approaching -1 we find that the active termination results in the reflection of a breather into a breather possibly followed by fluxons, antifluxons, or even breathers. This corresponds to an input of energy delivered by the termination. Similar results were obtained in Ref. 6 for the reflection of a single fluxon. Even in the case of a linear homogeneous wave equation we find a similar behavior, i.e., a wave of amplitude A is reflected into a wave of amplitude

$$A_{\text{refl}} = \frac{1-\alpha}{1+\alpha} A = \frac{R - (L/C)^{1/2}}{R + (L/C)^{1/2}} A, \tag{3.3}$$

or in energetic terms,

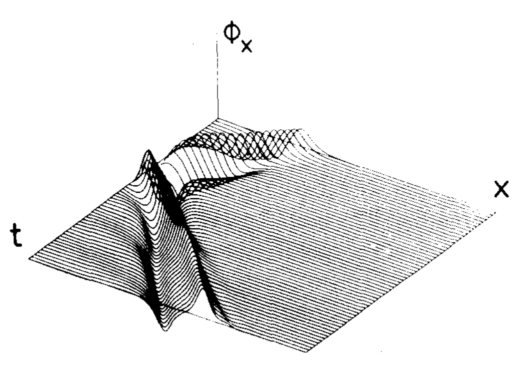


FIG. 5. Reflection of a breather at the left end of a finite line. Numerical solution of Eq. (3.1) with $l = 30$, $\alpha = -0.1$, $u = -0.5$, $\tan\theta = 2$, $x_0 = 7.5$, $t_0 = 0$, and $0 \leqslant t \leqslant 36$. At the active termination the breather is reflected into a breather of increased energy.

$$H_{\text{refl}} = \left(\frac{1-\alpha}{1+\alpha}\right)^2 H_{\text{inc}} = \left(\frac{\mathrm{R} - (\mathrm{L/C})^{1/2}}{\mathrm{R} + (\mathrm{L/C})^{1/2}}\right)^2 H_{\text{inc}}, \tag{3.4}$$

where H_{inc} is the energy of the incident wave and H_{refl} is the energy of the reflected wave. From a physical point of view the boundary conditions with $\alpha < 0$ is of minor importance (at least at the moment). Therefore we have omitted a detailed analysis of the active termination in this paper.

In this section we have illustrated by means of numerical computations how breathers (except for $\alpha \simeq 1$) are reflected into breathers when the boundary conditions (2.6) are chosen. In Sec. IV we examine the energetic properties of the breathers in detail.

IV. ENERGY CONSIDERATIONS

For the finite Josephson line we may introduce the dimensionless Hamiltonian (or total energy)

$$H = H(t) \equiv \int_0^l \left(\tfrac{1}{2}\phi_x^2 + \tfrac{1}{2}\phi_t^2 + 1 - \cos\phi\right) dx \tag{4.1}$$

Here the change of variables $H(2e/\hbar)^{3/2}(L/J_0)^{1/2} H$ has been introduced. Integrating by parts we get

$$\frac{dH}{dt} = \int_0^l \phi_t(\phi_{tt} - \phi_{xx} + \sin\phi)dx - \phi_x(0,t)\phi_t(0,t) + \phi_x(l,t)\phi_t(l,t). \tag{4.2}$$

Insertion of Eqs. (2.5) and (2.6) yields

$$\frac{dH}{dt} = -\alpha[\phi_t^2(0,t) + \phi_t^2(l,t)] \tag{4.3a}$$

or

$$\frac{dH}{dt} = -\frac{1}{\alpha}[\phi_x^2(0,t) + \phi_x^2(l,t)]. \tag{4.3b}$$

Thus the rate of change of the total energy of the system is equal to the effect absorbed or delivered by the determination component at $x = 0$. When $\phi_x(0,t) = 0$ and $\phi_x(l,t) = 0$, or $\phi_t(0,t) = 0$ and $\phi_t(l,t) = 0$, the total energy is constant.

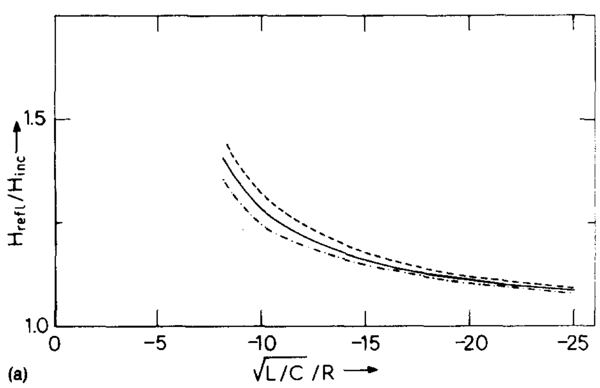
In order to investigate the influence of the termination components on the breather we have numerically integrated Eq. (4.1) before and after the reflection to obtain the energy of the incident breather and of the reflected breather H_{inc} and H_{refl}, respectively. For $x_0 > 0$ the energy of the incident breather is approximately equal to the energy of a breather on an infinite line

$$H_{inc} \simeq 16(1-\omega^2)^{1/2}/(1-u^2)^{1/2}. \quad (4.4)$$

In the examples throughout this paper H_{inc} differ less than 0.1% from the expression (4.4). Figures 7(a) and 7(b) show the results for three breathers of different frequency in terms of H_{refl}/H_{inc} vs $\alpha = (L/C)^{1/2}/R$ for $u = -0.5$, $x_0 = 7.5$, $t_0 = 0$, and $\tan\theta = 1$, 2, and 5. In Fig. 7(a) the parameter $\alpha < 0$ (active termination) while in Fig. 7(b) the parameter $\alpha > 0$ (passive termination). In Fig. 7(a) we find an energy increase. For α approaching -1 the energy input tends towards infinity.

Figure 7(b) shows that energy is absorbed by the passive termination. For $\alpha \simeq 1$ (i.e., the small signal characteristic impedance of the line is equal to the terminating resistance), approximately all incident energy is absorbed. The energy absorption models the practical use of a Josephson line, i.e., a microwave generator. The ratio H_{refl}/H_{inc} seems independent of variation in $\tan\theta$ for $\alpha \gg 1$ and $\alpha \simeq 0$. The results are in qualitative agreement with the linear theory mentioned above. Furthermore, it follows from the breather solution that

$$\phi_x + u\phi_t = -[4\psi/(1+\psi^2)](1-u^2)^{1/2} \times \sin\theta \tanh[\sin\theta\gamma(u)(x-ut-x_0)], \quad (4.5)$$



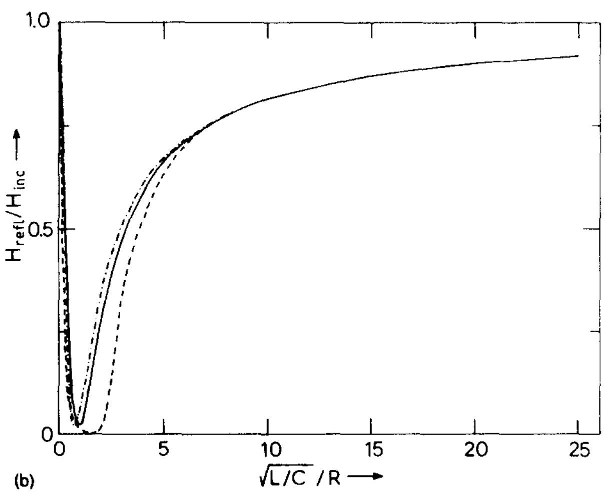
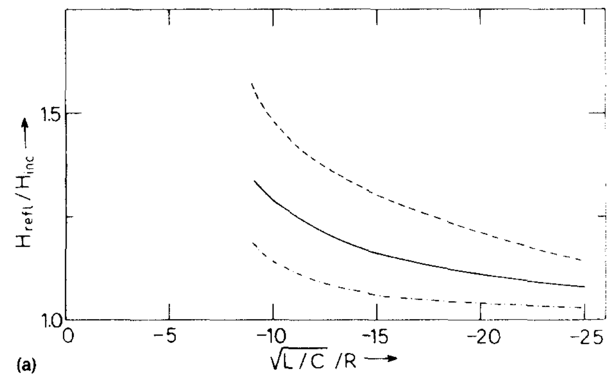


FIG. 7. The ratio H_{refl}/H_{inc} vs $\alpha = (L/C)^{1/2}/R$ for three breathers of different frequency. The parameters for the incident breather are chosen to be $u = -0.5$, $x_0 = 7.5$, $t_0 = 0$, and $\tan\theta = 1$ (dot-dash curve), $\tan\theta = 2$ (dashed curve), $\tan\theta = 5$ (solid curve). (a) shows the relative increase of energy as a result of the active termination. (b) shows the relative loss of energy which is transformed into em radiation.



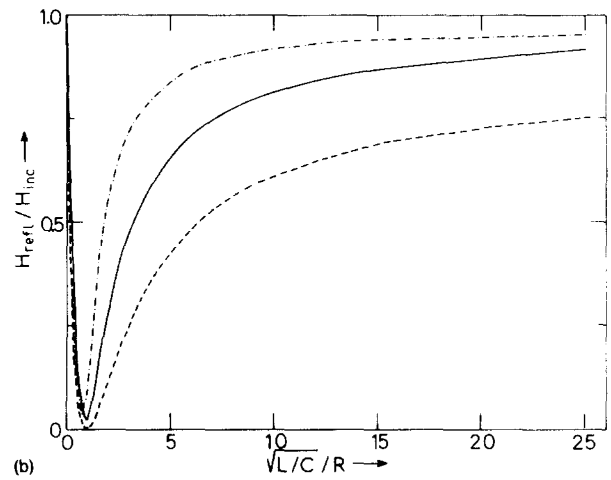


FIG. 8. The ratio H_{refl}/H_{inc} vs $\alpha = (L/C)^{1/2}/R$ for three breathers of different velocity. The parameters for the incident breather are chosen to be $u = -0.2$ (dot-dashed curve), $u = -0.5$ (solid curve), $u = -0.9$ (dashed curve), $x_0 = 7.5$, $t_0 = 0$, and $\tan\theta = 2$. (a) shows the relative increase of energy as a result of the active termination. (b) shows the relative loss of energy which is transformed into em radiation.

where ψ is the argument in the arctan function (3.2a). This expression shows matching for the condition (2.6) when $\alpha \simeq 1$ and $u \simeq -1$.

In Figs. 8(a) and 8(b), we show H_{refl}/H_{inc} vs $\alpha = (L/C)^{1/2}/R$ for three breathers of different initial velocity. The parameters for the incident breathers are chosen to be $u = -0.2$, -0.5, -0.9, and $x_0 = 7.5$, $t_0 = 0$, and $\tan\theta = 2$. The same qualitative behavior as in the former figures is observed. However, we find a dependence of H_{refl}/H_{inc} of the incident velocity for $\alpha \gg 1$ and $\alpha \simeq 0$. We have not been able to explain these highly nonlinear phenomena by means of any analytic approximation.

V. CONCLUSION

In the present paper we have examined the influence of a boundary on a breather. The boundary condition is chosen such that it models the effect of a passive or an active termination.

We find that an incident breather is reflected into a breather of less or increased energy. When the characteristic impedance of the line is equal to the terminating resistor ($\alpha \simeq 1$), the breather is almost annihilated. For an active termination ($\alpha < 0$) we find a breather reflected into a breather possibly followed by fluxons, antifluxons, or even breathers.

The results are in qualitative agreement with the linear theory.

Finally, we remark that it is unlikely to find breathers of constant velocity and frequency on a Josephson line because the presence of bias and loss will cause the breathers to collapse and eventually disappear. However, breathers will appear as an intermediate state when a fluxon and an antifluxon collapse into plasma oscillations.

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